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EXISTENCE OF SOLUTIONS FOR SEMILINEAR DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS IN BANACH SPACES

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Abstract. This paper is concerned with semilinear differential equations with nonlocal conditions in Banach spaces. Using the tools involving the measure of noncompactness and fixed point theory, existence of mild solutions is obtained without the assumption of compactness or equicontinuity on the associated linear semigroup.

1. Introduction and preliminaries

In this paper we discuss the semilinear differential equation with nonlocal condition

$$\frac{d}{dt}x(t) = Ax(t) + f(t, x(t)), \quad t \in (0, b], \tag{1.1}$$

$$x(0) = x_0 + g(x) \tag{1.2}$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{T(t) : t \geq 0\}$ of linear operators defined on a Banach space X, $f : [0, b] \times X \to X$ and $g : C([0, b]; X) \to X$ are appropriate given functions.

The theory of differential equations with nonlocal conditions was initiated by Byszewski and it has been extensively studied in the literature. We infer to some of the papers below. Byszewski ([3, 4]), Byszewski and Lakshmikantham ([6]) give the existence and uniqueness of mild solutions and classical solutions when g and f satisfy

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Lipschitz-type conditions with the special type of g. In [5], Byszewski and Akca give the existence of semilinear functional differential equation when $T(t)$ is compact, and g is convex and compact on a given ball of $C([0, b]; X)$. Dong and Li [8, 9] discussed viable domain for semilinear functional differential equation when $T(t)$ is compact. Ntouyas and Tsamatos [14] studied the existence for semilinear evolution equations with nonlocal conditions. Xue [16] proved the existence results for nonlinear nonlocal Cauchy problem. In [17], Xue discussed the semilinear case when f and g are compact and when g is Lipschitz and $T(t)$ is compact. Fan et al. [7], Guedda[10] and Xue [18] studied some semilinear equations under the conditions in respect of the measure of noncompactness.

In this paper, by using the tools involving the measure of noncompactness and fixed point theory, we obtain existence of mild solution of semilinear differential equation with nonlocal conditions (1.1)-(1.2), and the compactness of solution set, without the assumption of compactness or equicontinuity on the associated semigroup. Our results extend and improve the correspondence results in [3, 4, 5, 6, 17]. We indicate that the method we used in this paper is different from that in [7] or [10].

Throughout this paper X will represent a Banach space with norm $\|\cdot\|$. As usual, $C([a, b]; X)$ denotes the Banach space of all continuous X-valued functions defined on $[a, b]$ with norm $\|x\|_{[a,b]} = \sup_{s\in[a,b]} \|x(s)\|$ for $x \in C([a, b]; X)$.

Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a strongly continuous semigroup $\{T(t) : t \geq 0\}$ of linear operators on X. We always assume that $\|T(t)\| \leq M$ $(M \geq 1)$ for every $t \in [0, b]$.

For more details of the semigroup theory we refer the readers to [15].

2. Measure of noncompactness

In this section we recall the concept of the measure of noncompactness in Banach spaces.

Definition 2.1. Let E^+ be a positive cone of an ordered Banach space $(E, \leq)$. A function Φ on the collection of all bounded subsets of a Banach space X with values in E^+ is called a measure of noncompactness if $\Phi(\overline{co}B) = \Phi(B)$ for all bounded subset $B \subset X$, where $\overline{co}B$ stands for the closed convex hull of B.

A measure of noncompactness Φ is said to be:

(i) *monotone* if for all bounded subset B_1, B_2 of X, $B_1 \subset B_2$ implies $\Phi(B_1) \leq \Phi(B_2)$;

(ii) *nonsingular* if $\Phi(\{a\} \cup B) = \Phi(B)$ for every $a \in X$ and every nonempty subset $B \subset X$;

(iii) *regular* if $\Phi(B) = 0$ if and only if B is relatively compact in X.

One of the most important examples of measure of noncompactness is the $Hausdorff's\ measure\ of\ noncompactness$ β_Y, which is defined by

$$\begin{aligned} \beta_Y(B) &= \inf\{r > 0; B \textit{ can be covered with a finite} \\ &\qquad \textit{number of balls of radii smaller then } r\} \end{aligned}$$

for bounded set B in a Banach space Y.

The following properties of Hausdorff's measure of noncompactness are well known:

Lemma 2.2. *([2]): Let Y be a real Banach space and $B, C \subseteq Y$ be bounded,the following properties are satisfied :*

(1). B is pre-compact if and only if $\beta_Y(B) = 0$;

(2). $\beta_Y(B) = \beta_Y(\overline{B}) = \beta_Y(convB)$ where $\overline{B}$ and $convB$ mean the closure and convex hull of B respectively;

(3). $\beta_Y(B) \leq \beta_Y(C)$ when $B \subseteq C$;

(4). $\beta_Y(B + C) \leq \beta_Y(B) + \beta_Y(C)$ where $B + C = \{x + y; x \in B, y \in C\}$;

(5). $\beta_Y(B \cup C) \leq max\{\beta_Y(B), \beta_Y(C)\}$;

(6). $\beta_Y(\lambda B) = |\lambda|\beta_Y(B)$ for any $\lambda \in R$;

(7). If the map $Q : D(Q) \subseteq Y \to Z$ is Lipschitz continuous with constant k then $\beta_Z(QB) \leq k\beta_Y(B)$ for any bounded subset $B \subseteq D(Q)$,where Z be a Banach space;

(8). $\beta_Y(B) = \inf\{d_Y(B, C); C \subseteq Y$ *be precompact*$\} = \inf\{d_Y(B, C); C \subseteq Y$ *be finite valued*$\}$, *where* $d_Y(B, C)$ *means the nonsymmetric (or symmetric) Hausdorff distance between* B *and* C *in* Y.

(9). If $\{W_n\}_{n=1}^{+\infty}$ *is a decreasing sequence of bounded closed nonempty subsets of* Y *and* $\lim_{n\to+\infty} \beta_Y(W_n) = 0$, *then* $\bigcap_{n=1}^{+\infty} W_n$ *is nonempty and compact in* Y.

The map $Q : W \subseteq Y \to Y$ is said to be a β_Y- *contraction* if there exists a positive constant $k < 1$ such that $\beta_Y(Q(B)) \leq k\beta_Y(B)$ for any bounded closed subset $B \subseteq W$, where Y is a Banach space.

Lemma 2.3. *([12]): Let* $W \subset Y$ *is bounded closed and convex and* $Q : W \to W$ *is a continuous* β_Y-*contraction. If the fixed point set of* Q *is bounded, then it is compact.*

In this paper we denote by β the $Hausdorff's$ measure of noncompactness of X and denote by β_c the $Hausdorff's$ measure of noncompactness of $C([a, b]; X)$. To discuss the existence we need the following lemmas in this paper.

Lemma 2.4. *([2]): If* $W \subseteq C([a, b]; X)$ *is bounded, then*

$$\beta(W(t)) \leq \beta_c(W)$$

for all $t \in [a, b]$, *where* $W(t) = \{u(t); u \in W\} \subseteq X$.*Furthermore if* W *is equicontinuous on [a,b], then* $\beta(W(t))$ *is continuous on* $[a, b]$ *and*

$$\beta_c(W) = \sup\{\beta(W(t)), t \in [a, b]\}.$$

Lemma 2.5. *([11, 12]): If* $\{u_n\}_{n=1}^{\infty} \subset L^1(a, b; X)$ *is uniformly integrable, then* $\beta(\{u_n(t)\}_{n=1}^{\infty})$ *is measurable and*

$$\beta(\{\int_a^t u_n(s)ds\}_{n=1}^{\infty}) \leq 2\int_a^t \beta(\{u_n(s)\}_{n=1}^{\infty})ds. \tag{2.1}$$

Now we consider another measure of noncompactness in the Banach space $C([a, b]; X)$. For a bounded subset $B \in C([a, b]; X)$, we define

$$\chi_1(B) = \sup_{t\in[a,b]} \beta(B(t)),$$

then χ_1 is well defined from the properties of Hausdorff's measure of noncompactness. We also define

$$\chi_2(B) = \sup_{t\in[a,b]} mod_C(B(t)),$$

where $mod_C(B(t))$ is the modulus of equicontinuity of the set of functions B at point t given by the formula

$$mod_C(B(t)) = \lim_{\delta\to 0}\sup\{\|x(t_1) - x(t_2)\| : t_1, t_2 \in (t-\delta, t+\delta), x \in B\}.$$

Define

$$\chi(B) = \chi_1(B) + \chi_2(B).$$

Then χ is a monotone and nonsingular measure of noncompactness in the space $C([a, b]; X)$. Further, χ is also regular by the famous Ascoli-Arzela's theorem. Similar definitions with χ_1, χ_2 and χ can be found in [2].

The following property of regular measure of noncompactness is useful for our results

Lemma 2.6. *Suppose that Φ is a regular measure of noncompactness in a Banach space Y, and $\{B_n\}$ is a sequence of nonempty, closed and bounded subsets in Y satisfying $B_{n+1} \subset B_n$ for $n = 1, 2, \cdots$. If $\lim_{n\to\infty} \Phi(B_n) = 0$, $B = \cap_{n\geq 1} B_n \neq \emptyset$ and B is a compact subset in Y.*

3. The existence of mild solution

In order to define the concept of mild solution for (1.1)-(1.2), by comparison with the abstract Cauchy initial value problem

$$\frac{d}{dt}x(t) = Ax(t) + f(t), \quad x(0) = \overline{x} \in X,$$

whose properties are well known [15], we associate (1.1)-(1.2) to the integral equation

$$x(t) = T(t)(x_0 + g(x)) + \int_0^t T(t-s)f(s, x(s))ds, \quad t \in [0, b]. \tag{3.1}$$

Definition 3.1. A continuous function $x : [0, b] \to X$ is said to be a mild solution to the nonlocal problem (1.1)-(1.2) if $x(0) = x_0 + g(x)$ and (3.1) is satisfied.

In this section by using the usual techniques of the measure of noncompactnes and its applications in differential equations in Banach spaces (see, e.g. [2], [13]) we give some existence results for the nonlocal problem (1.1)-(1.2). Here we list the following hypotheses.

(Hf):(1) $f : [0, b] \times X \to X$ satisfies the *Carathéodory*-type condition, i.e.,$f(\cdot, x) : [0, b] \to X$ is measurable for all $x \in X$ and $f(t, \cdot) : X \to X$ is continuous for a.e. $t \in [0, b]$;

(2): There exists a function $h : [0, b] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(\cdot, s) \in L(0, b; \mathbb{R}^+)$ for every $s \geq 0$, $h(t, \cdot)$ is continuous and increasing for a.e. $t \in [0, b]$, and $\|f(t, x)\| \leq h(t, \|x\|)$ for a.e. $t \in [0, b]$ and all $x \in X$, and for each positive constant K, the following scalar equation

$$m(t) = MK + M \int_0^t h(s, m(s))ds, \quad t \in [0, b] \tag{3.2}$$

has at least one solution;

(3):There exists $\eta \in L(0, b; \mathbb{R}^+)$ such that:

$$\beta(T(s)f(t, D)) \leq \eta(t)\beta(D) \tag{3.3}$$

for a.e. $t, s \in [0, b]$ and any bounded subset $D \subset X$.

(Hg): (1) $g : C([0, b]; X) \to X$ is continuous and compact;

(2) There exists a constant $N > 0$ such that

$$\|g(x)\| \leq N \tag{3.4}$$

for all $x \in X$.

Remark 3.2. If the semigroup $\{T(t) : t \geq 0\}$ or the function f is compact (see, e.g.,[5, 17]), or f satisfies Lipschitz-type condition (see, e.g., [3, 4]), then (Hf)(3) is automatically satisfied.

Now, we give an existence result under the above hypotheses.

Theorem 3.3. *Assume the hypotheses (Hf) and (Hg) are satisfied. Then for each $x_0 \in X$, the solution set of the problem (1.1)-(1.2) is a nonempty compact subset of the space $C([0, b]; X)$.*

Proof. Let $m : [0, b] \to \mathbb{R}^+$ be a solution of the scalar equation:

$$m(t) = M(\|x_0\| + N) + M \int_0^t h(s, m(s))ds, \quad t \in [0, b]. \tag{3.5}$$

Define a map $\Gamma : C([0,b];X) \to C([0,b];X)$ by

$$\Gamma x(t) = T(t)(x_0 + g(x)) + \int_0^t T(t-s)f(s,x(s))ds, \quad t \in [0,b]$$

for all $x \in C([0,b]:X)$. It is easily seen that $x \in C([0,b];X)$ is a mild solution of problem (1.1)-(1.2) if and only if x is a fixed point of Γ. We shall show that Γ has a fixed point by Schauder's fixed point theorem. To do this, we first see that Γ is continuous by the usual technique involving (Hf), (Hg) and Lebesgue's dominated convergence theorem.

We denote by $W_0 = \{x \in C([0,b];X) : \|x(t)\| \leq m(t), \forall t \in [0,b]\}$. Then $W_0 \subset C([0,b];X)$ is bounded and convex.

Define $W_1 = \overline{conv}\Gamma W_0$, where $\overline{conv}$ means the closure of the convex hull in $C([0,b];X)$. Then it is easily seen that $W_1 \subset C([0,b];X)$ is closed and convex. Furthermore, for every $x \in C([0,b];X)$, we have

$$\|\Gamma x(t)\| \leq M(\|x_0\| + N) + M\int_0^t h(s,m(s))ds = m(t)$$

for $t \in [0,b]$. Hence $\Gamma x \in W_0$ for all $x \in W_0$. It then follows that W_1 is bounded and $W_1 \subset W_0$.

Define $W_{n+1} = \overline{conv}\Gamma W_n$ for $n = 1,2,\cdots$. From the above proof we have that $\{W_n\}_{n=1}^{\infty}$ is an decreasing sequence of bounded closed, convex and nonempty subsets in $C([0,b];X)$.

Now, for every $n \geq 1$ and $t \in (0,b]$, $W_n(t)$ and $\Gamma W_n(t)$ are bounded subsets of X. Hence for any $\varepsilon > 0$, there is a sequence $\{x_k\}_{k=1}^{\infty} \subset W_n$ such that (see, e.g., [1], pp.125)

$$\begin{aligned}\beta(\Gamma W_n(t)) &\leq 2\beta(\{\Gamma x_k(t)\}_{k=1}^{\infty}) + \varepsilon \\ &\leq 2\beta(T(t)(x_0 + g(\{x_k\}_{k=1}^{\infty}) \\ &\quad +2\beta(\int_0^t T(t-s)f(s,\{x_k(s)\}_{k=1}^{\infty})ds) + \varepsilon.\end{aligned}$$

From the compactness of g, Lemma 2.2, Lemma 2.5 and (Hf)(3), we have

$$\begin{aligned}\beta(W_{n+1}(t)) &= \beta(\Gamma W_n(t)) \\ &\leq 2\beta(\int_0^t T(t-s)f(s,\{x_k(s)\}_{k=1}^{\infty})ds)+\varepsilon \\ &\leq 2\int_0^t \beta(T(t-s)f(s,\{x_k(s)\}_{k=1}^{\infty}))ds+\varepsilon \\ &\leq 2\int_0^t \eta(s)\beta(\{x_k(s)\}_{k=1}^{\infty})ds+\varepsilon \\ &\leq 2\int_0^t \eta(s)\beta(W_n(s))ds+\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows from the above inequality that

$$\beta(W_{n+1}(t)) \leq 2\int_0^t \eta(s)\beta(W_n(s))ds \tag{3.6}$$

for all $t \in (0,b]$. Define functions $f_n : [0,b] \to [0,+\infty)$ by

$$f_n(t) = \beta(W_n(t)),$$

then we obtain from (3.6) that

$$f_{n+1}(t) \leq 2\int_0^t \eta(s)f_n(s)ds \tag{3.7}$$

for all $t \in [0,b]$.

Notice that $\chi_1(W_n) = \sup_{0\leq t\leq b}\beta(W_n(t)) = \sup_{0\leq t\leq b} f_n(t)$. Denote $\varepsilon_n = \chi_1(W_n)$, then $\beta(W_n(t)) \leq \varepsilon_n$ for all $t \in [0,b]$. Now we consider $\chi_2(W_n)$. We claim that there is a constant K_1 such that $\chi_2(W_n) \leq K_1\varepsilon_n$. To do this, it is sufficient to prove that for every $t_0 \in [0,b]$, $mod_C(W_n(t_0)) \leq K_1\varepsilon_n$.

Take $\varepsilon > 0$ arbitrary. First note that, since $f(s,(W_n)(s))$ is uniformly integrable on $[0,b]$, there is $\delta_1 > 0$ such that

$$\|\int_{\tau_1}^{\tau_2} T(t-s)f(s,x(s))ds\| < \varepsilon \tag{3.8}$$

for $0 < \tau_2 - \tau_1 < 2\delta_1$, $0 \leq s \leq t \leq b$ and every $x \in W_n$. For every $y \in \Gamma(W_{n-1})$, there is an $x \in W_{n-1}$ such that for all $t \in [0,b]$,

$$y(t) = T(t)(x_0+g(x)) + \int_0^t T(t-s)f(s,x(s))ds.$$

If $0 < t' < t$ then we have

$$\begin{aligned} y(t) &= T(t-t')T(t')(x_0 + g(x)) + T(t-t')\int_0^{t'} T(t'-s)f(s,x(s))ds \\ &\quad + \int_{t'}^{t} T(t-s)f(s,x(s))ds \\ &= T(t-t')y(t') + \int_{t'}^{t} T(t-s)f(s,x(s))ds. \end{aligned}$$

It follows that for any $t_1, t_2 \in (t_0 - \delta_1, t_0 + \delta_1)$, (we may assume that $t_0 - \delta_1 > 0$),

$$y(t_1) = T(t_1 - t_0 + \delta_1)y(t_0 - \delta_1) + \int_{t_0-\delta_1}^{t_1} T(t_1 - s)f(s,x(s))ds, \tag{3.9}$$

$$y(t_2) = T(t_2 - t_0 + \delta_1)y(t_0 - \delta_1) + \int_{t_0-\delta_1}^{t_2} T(t_2 - s)f(s,x(s))ds. \tag{3.10}$$

Now, on the basis of the definition of Hausdorff's measure of non-compactness and the fact that $\beta(W_n(t_0 - \delta_1)) \leq \varepsilon_n$, we may find $y_1, y_2, \cdots, y_k$, such that

$$W_n(t_0 - \delta_1) \subset \bigcup_{i=1}^{k} B(y_i(t_0 - \delta_1), 2\varepsilon_n),$$

where $B(u, r)$ denote the ball centered at u and radius r. Hence there is an i, $1 \leq i \leq k$ such that

$$\|y(t_0 - \delta_1) - y_i(t_0 - \delta_1)\| < 2\varepsilon_n. \tag{3.11}$$

On the other hand, on account of the strong continuity of $T(\cdot)$, there is a $\delta > 0$ (we may choose $\delta < \delta_1$) such that

$$\|T(\tau)y_i(t_0 - \delta_1) - y_i(t_0 - \delta_1)\| < \varepsilon \tag{3.12}$$

for all $\tau \in (0, \delta)$ and $i = 1, 2, \cdots, k$. From (3.8)-(3.12), we obtain that

$$\begin{aligned} \|y(t_1) - y(t_2)\| &\leq \|T(t_1 - t_0 + \delta_1)y(t_0 - \delta_1) - y(t_0 - \delta_1)\| \\ &\quad + 2\|y(t_0 - \delta_1) - y_i(t_0 - \delta_1)\| \\ &\quad + \|T(t_2 - t_0 + \delta_1)y(t_0 - \delta_1) - y(t_0 - \delta_1)\| \\ &\quad + \|\int_{t_0-\delta_1}^{t_1} T(t_1 - s)f(s, x_s)ds - \int_{t_0-\delta_1}^{t_2} T(t_2 - s)f(s, x_s)ds\| \\ &\leq 4\varepsilon_n + 4\varepsilon \end{aligned}$$

for $t_1, t_2 \in (t_0 - \delta, t_0 + \delta)$. Since $\varepsilon > 0$ is taken arbitrary, we get that

$$\|y(t_1) - y(t_2)\| \leq 4\varepsilon_n$$

for $t_1, t_2 \in (t_0 - \delta, t_0 + \delta)$, which implies that

$$mod_C(W_n(t_0)) \leq 4\varepsilon_n$$

for every $t_0 \in [0, b]$. Accordingly,

$$\chi_2(W_n) \leq 4\varepsilon_n = 4\chi_1(W_n). \tag{3.13}$$

Coming back to consider f_n, since W_n is decreasing for n, we know that $f(t) = \lim_{n\to\infty} f_n(t)$ exists for $t \in [0, b]$. Taking limit as $n \to \infty$ in (3.7), we have

$$f(t) \leq 2\int_0^t \eta(s)f(s)ds, \tag{3.14}$$

for $t \in [0, b]$. It then follows that $f(t) = 0$ for all $t \in [0, b]$. This means that $\lim_{n\to\infty} \chi_1(W_n) = 0$. The inequality (3.13) implies that $\lim_{n\to\infty} \chi_2(W_n) = 0$, and hence $\lim_{n\to\infty} \chi(W_n) = 0$. Using Lemma 2.6 we know that $W = \cap_{n=1}^{\infty} W_n$ is convex compact and nonempty in $C([0, b]; X)$ and $\Gamma W \subset W$. By the famous Schauder's fixed point theorem, there exists at least one fixed point $x \in W$ of Γ, which is the mild solution of (1.1)-(1.2). From the proof we can also see that all the fixed points of Γ are in W which is compact in $C([0, b]; X)$. The continuity of the map Γ implies the closeness of the fixed point set. Hence we conclude that the solution set of problem (1.1)-(1.2) is compact in the space $C([0, b]; X)$. □

Since $\{T(t)\}$ is a C_0-semigroup, condition (Hf)(3) can be replaced by

(Hf)($3'$): There exists $\tilde{\eta} \in L(0, b; \mathbb{R}^+)$ such that:

$$\beta(f(t, D)) \leq \tilde{\eta}(t)/M \cdot \beta(D) \tag{3.15}$$

for a.e. $t, s \in [0, b]$ and any bounded subset $D \subset X$.

From Theorem 3.3, we can get the following obvious result:

Theorem 3.4. *Assume the hypotheses (Hf)(1)(2)($3'$) and (Hg) are satisfied. Then for each $x_0 \in X$, the solution set of problem (1.1)-(1.2) is a nonempty compact subset of the space $C([0, b]; X)$.*

In some of the early related results in references and the two results above, it is supposed that the map g is uniformly bounded. We indicate here that this condition can be released. Indeed, the fact that g is compact implies that g is bounded on bounded subset. And the hypothesis (Hf)(2) may be difficult to be verified sometimes. Here we give an existence result under another growth condition of f when g is not uniformly bounded. Precisely, we replace the hypothesis (Hf)(2) by

(Hf)($2'$): There exists a function $\alpha \in L(0, b; \mathbb{R}^+)$ and an increasing function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t, v)\| \leq \alpha(t)\Omega(\|v\|)$$

for a.e. $t \in [0, b]$ and all $v \in X$.

Theorem 3.5. *Suppose that the hypotheses (Hf)(1)($2'$)(3) and (Hg)(1) are satisfied. If*

$$\limsup_{k\to\infty} \frac{M}{k}(\gamma(k) + \Omega(k)\int_0^b \alpha(s)ds < 1, \tag{3.16}$$

where $\gamma(k) = \sup\{\|g(x)\| : \|x\| \leq k\}$*, then for each* $x_0 \in X$*, the solution set of problem (1.1)-(1.2) is a nonempty compact subset of the space* $C([0, b]; X)$*.*

Proof. The inequality (3.16) implies that there exists a constant $k > 0$ such that

$$M(\|x\| + \gamma(k) + \Omega(k)\int_0^b \alpha(s)ds) < k.$$

As in the proof of Theorem 3.3, let $W_0 = \{x \in X : \|x\| \leq k\}$ and $W_1 = \overline{conv}\Gamma W_0$. Then for any $x \in W_1$, we have

$$\|x(t)\| \leq M(\|x\| + \gamma(k)) + M\Omega(k)\int_0^b \alpha(s)ds < k$$

for $t \in [0, b]$. It follows that $W_1 \subset W_0$. So we can complete the proof similarly to Theorem 3.3. □

Remark 3.6. In some previous papers the authors assumed that the space X is a separable Banach space and the semigroup $T(t)$ is equicontinuous (see, e.g., [17, 18]). We mention here that these assumptions are not necessary.

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